Fundamental Theorem of Line Integrals (Section 16.2)

Suppose a surface

$z = f(x, y)$ is defined

over a region R in $\mathbb{R}^2$

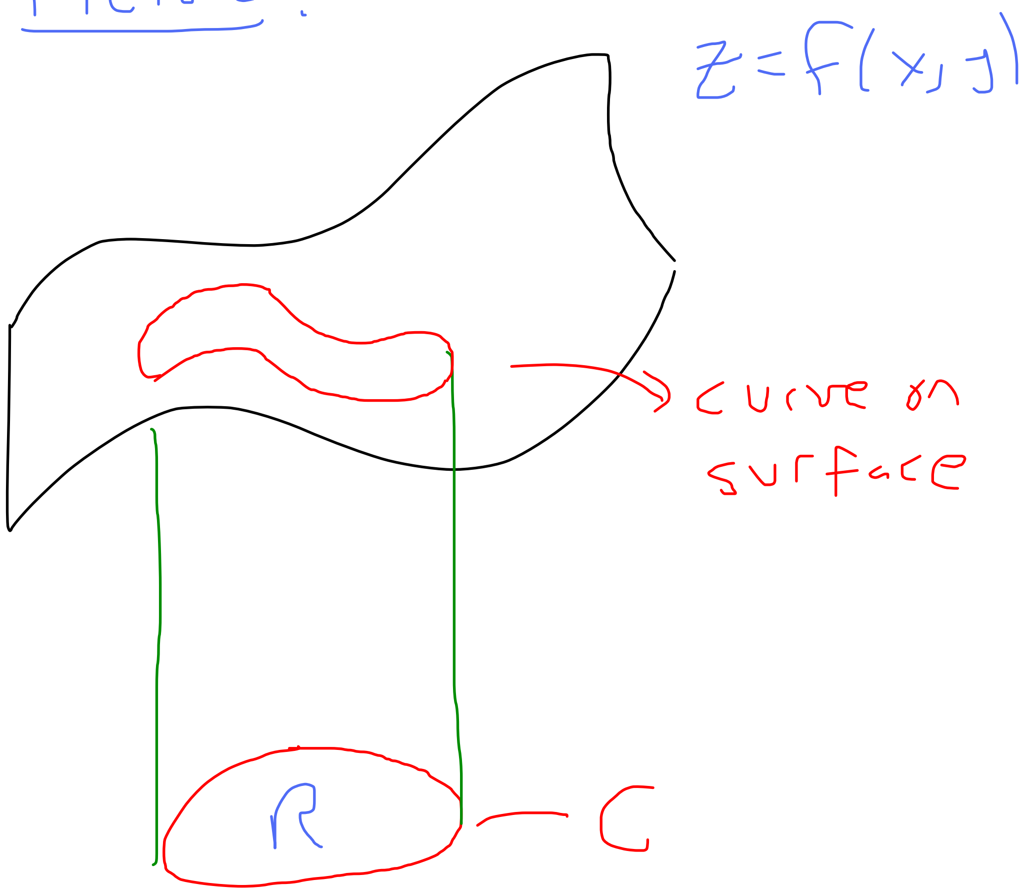
with boundary C. If

f is also defined on C,

the graph of these points

gives a "curve on the surface"

Picture.



Suppose we can parameterize C by $r(t) = \langle x(t), y(t) \rangle$ from $t = a$ to $t = b$ and that C is traced out no more than once on $[a, b]$

Suppose r' is continuous on $[a, b]$ and $\|r'\| > 0$

The area under the graph of $z = f(x,y)$ and over C is given by the line integral

$$\boxed{\int_a^b f(x(t),y(t))\sqrt{(x'(t))^2+(y'(t))^2}\,dt}$$

This is sometimes denoted by

$$\int_C f(x,y)\,ds$$ (s = arc length).

Notice: If f is constantly one on C, we recover the arclength of C – as we should!

Example 1: Find the line integral of $f(x,y) = x + y^2$ over $C = \{(x,y) : x^2 + y^2 = 1\}$.

Parameterize C by

$$r(t) = \langle \cos(t), \sin(t) \rangle,$$

$$0 \leq t < 2\pi.$$

Then $x(t) = \cos(t)$, $y(t) = \sin(t)$,

$x'(t) = -\sin(t)$, $y'(t) = \cos(t)$,

so $\sqrt{(x'(t))^2 + (y'(t))^2}$

$= \sqrt{\sin^2(t) + \cos^2(t)}$

$= 1$.

Using the formula,

$$\int_C x + y^2 \, ds$$

$$= \int_0^{2\pi} \left(\cos(t) + \sin^2(t)\right) dt$$

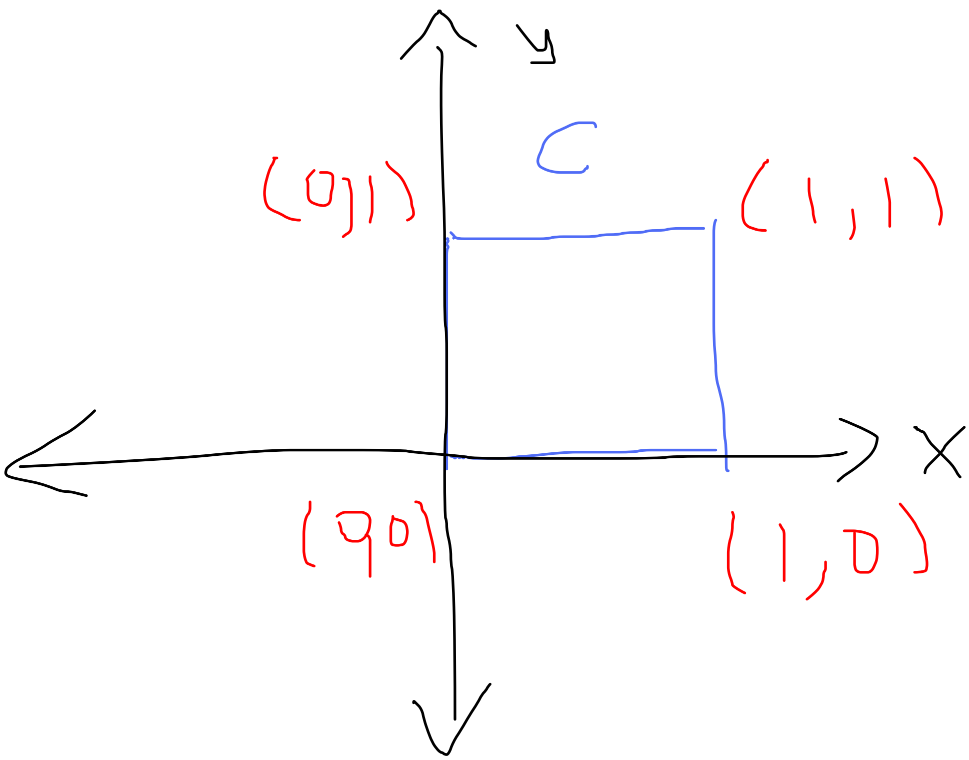
$$= \int_0^{2\pi} \left(\cos(t) + \frac{1-\cos(2t)}{2}\right) dt$$

$$= \left(\sin(t) + \frac{t}{2} - \frac{\sin(2t)}{4}\right) \Big|_0^{2\pi}$$

$$= \boxed{\pi}$$

We can relax our requirements for f by allowing finitely many points where f' does not exist.

Example 2: C is the square with vertices $(0,0)$ $(0,1)$, $(1,0)$, and $(1,1)$.



There is no single parameterization for C, but we can do it in four:

$$r(t) = \begin{cases} \langle 0, t \rangle, & 0 \le t \le 1 \\ \langle t-1, 1 \rangle, & 1 \le t \le 2 \\ \langle 1, 3-t \rangle, & 2 \le t \le 3 \\ \langle 4-t, 0 \rangle, & 3 \le t \le 4 \end{cases}$$

Or you could write

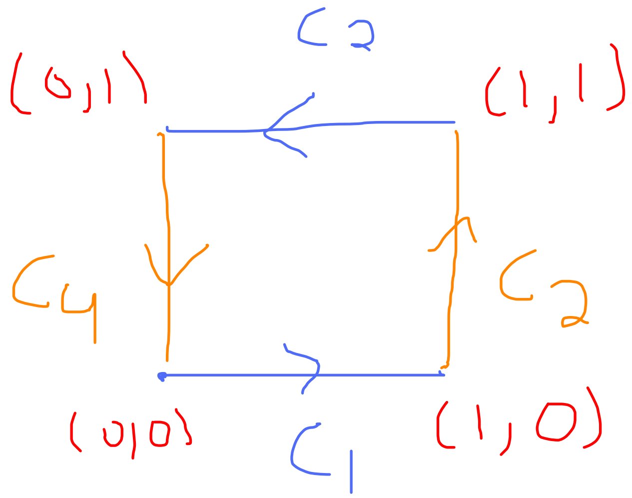
C in 4 pieces

C_1: $r_1(t) = \langle 0, t \rangle$, $0 \le t \le 1$

C_2: $r_2(t) = \langle t, 1 \rangle$, $0 \le t \le 1$

C_3: $r_3(t) = \langle 1, 1-t \rangle$, $0 \le t \le 1$

C_4: $r_4(t) = \langle 1-t, 0 \rangle$, $0 \le t \le 1$



Integrate F over C by

$$\int_C f(x,y)\,ds$$

$$= \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds$$

$$+ \int_{C_3} f(x,y)\,ds + \int_{C_4} f(x,y)\,ds$$

We can also integrate by holding x or y constant on C:

if C is parameterized by $\langle x(t), y(t)\rangle$ for $a \le t \le b$, define

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

It is not true in general that $\int_C f(x,y)\,dx + \int_C f(x,y)\,dy$

$$= \int_C f(x,y)\,ds$$

Strange Notation: (unfortunately standard)

$$\int_C f(x,y)\,dx + \int_C g(x,y)\,dy$$

$$= \int_C f(x,y)\,dx + g(x,y)\,dy$$

Remember that $dx = x'(t)\,dt$ and $dy = y'(t)\,dt$, so this odd notation actually makes sense.

Even terser shorthand:

$$\int_C f\,dx + g\,dy$$

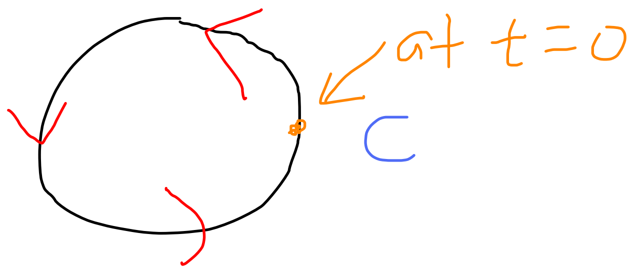
Orientation for Line Integrals

An orientation of a curve C determines a direction along which to travel on a curve.

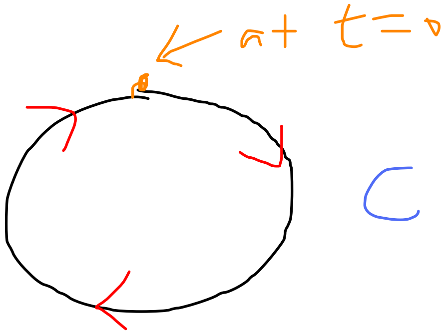
Example 3: $C = \{(x,y) : x^2+y^2=1\}$

One orientation (counterclockwise)

$r_1(t) = \langle \cos(t), \sin(t) \rangle$

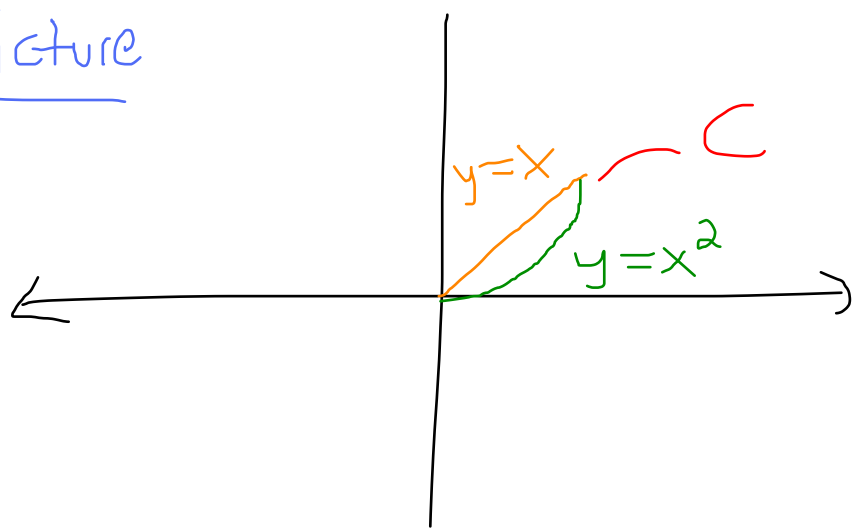


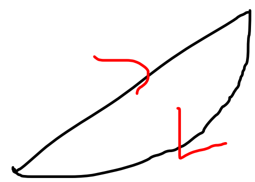
$r_2(t) = \langle \sin(t), \cos(t) \rangle$



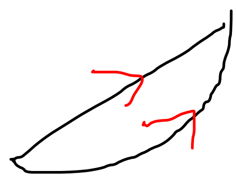
Example 4: Find all orientations on the curve C determined by the graphs of $y=x$ and $y=x^2$ for $0 \le x \le 1$.

Picture





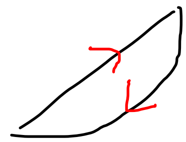
is allowed, but



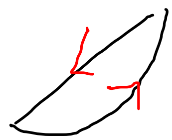
is not -

you don't know which direction to go at (1,1) or (0,0)!

Only



and



Given a curve C with an orientation, call C with the opposite orientation $-C$

If C is parameterized by $\langle x(t), y(t) \rangle$ from $t=a$ to $t=b$, the opposite orientation is given by

$$\boxed{\langle x(a-t+b), y(a-t+b) \rangle}$$

Q. Does the value of the line integral depend on the parameterization of C?

One can check that

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$$

In fact, as long as C is traced out no more than once for any two given parameterizations, then the value of the line integral is independent of parameterization

But by the chain rule,

$$\int_{-C} f(x,y)\,dx = -\int_{C} f(x,y)\,dx$$

and similarly,

$$\int_{-C} f(x,y)\,dy = -\int_{C} f(x,y)\,dy$$